EXERCISESET 7, TOPOLOGY IN PHYSICS

- The hand-in exercises are the exercise 2 & 3.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday April 17, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: The Pfaffian. Let us write G_n for the Grassmann algebra on n-variables θ^i, $i = 1, \ldots, n$. Define the *Berezin integral* $T : G_n \to \mathbb{R} \subset G_n$ by

$$T(\theta^1 \cdots \theta^n) := 1,$$

while T vanishes on products of degree $\leq n-1$.

a) Show that T equals $\int d\theta^1 \cdots \int d\theta^n = I_n \circ I_{n-1} \circ \ldots \circ I_1$ where I_i denotes the integral over θ_i.

b) Suppose that n is even. Given a skew-symmetric $n \times n$-matrix A, define its *Pfaffian* by

$$\mathrm{Pf}(A) := T\left(\exp \frac{1}{2} \sum_{i,j} A_{ij} \theta^i \theta^j\right),$$

where exp is defined in G_n by its power series (which terminates after finitely many terms). Show that

$$\mathrm{Pf}(A)^2 = \det(A).$$

(*Hint: Recall from the previous lecture (notes) how the integrals over θ_i behave under substitution of variables.*)

c) Show that the Pfaffian defines a GL^+-invariant polynomial of degree $n/2$, i.e. show that

$$\mathrm{Pf}(gAg^{-1}) = \mathrm{Pf}(A)$$

for all $g \in GL(n, \mathbb{R})$ such that $\det(g) > 0$.

Remark. Because of property c) above, one can use the Pfaffian to define a characteristic class of an even-dimensional oriented manifold M, called the *Euler class*, as follows: The curvature R of a riemannian metric on M is a skew-symmetric 2-form, so we can apply the Chern–Weil construction to define the cohomology class

$$e(M) := [\mathrm{Pf}(R)] \in H^{\dim(M)}_{\mathrm{dR}}(M).$$

$\star$ **Exercise 2: Clifford algebras and Grassmann variables.** The Clifford algebra and Grassmann variables may look similar, they are not the same: notice that in the Clifford algebra we have $\psi_i^2 = \pm 1$, whereas in the Grassmann algebra we have $\theta_i^2 = 0$. There is a relation however between the two, and the purpose of this exercise is to explore this connection. We will consider the general Clifford algebra $\mathrm{Cliff}_{p,q}$ and put $n := p + q$

a) Show that both $\mathrm{Cliff}_{p,q}$ and the Grassmann algebra on n-variables are of dimension 2^n.
b) In the Grassmann algebra on n-variables $\theta_i,\ i = 1, \dots, n$ introduce the operators
$$\hat{\psi}_i := \theta_i \pm \frac{d}{d\theta_i},$$
with the $-$-sign for $i = 1, \dots, p$ and $+$ for $i = p + 1, \dots, n$. Show that the $\hat{\psi}_i$ satisfy the commutation relations of the Clifford algebra $\mathrm{Cliff}_{p,q}$.

$\star$ **Exercise 3: Chirality.** Consider the Clifford algebra $\mathrm{Cliff}_{p,q}$ and write $n := p + q$. Define the volume element
$$\tau := \psi_1 \cdots \psi_n.$$

a) Show that
$$\tau^2 = (-1)^{\frac{n(n-1)}{2} + p}, \quad \psi_i \tau = (-1)^{n-1} \tau \psi_i$$
b) Suppose that $\tau^2 = -1$ (for example in $\mathrm{Cliff}_{3,1}$). Show that
$$\pi^{\pm} := \frac{1 \pm i\tau}{2}$$
satisfy
$$\pi^+ + \pi^- = 1, \quad [\pi^+, \pi^-] = 0, \quad (\pi^{\pm})^2 = \pi^{\pm}$$

Exercise 4: The Euler–Dirac operator. In this exercise we turn to geometry. Let (M, g) be a compact Riemannian manifold. Using the Riemannian metric, we can identify TM with T^*M. Taking sections, this means that we can map vector fields to differential 1-forms and vice versa: we write $\tilde{X}$ for the 1-form associated to a vector field X. In local coordinates we have $\tilde{X}_j = X^i g_{ij}$. We write $\mathrm{Cliff}(TM)$ for the bundle of Clifford algebras. We consider the vector bundle $\bigwedge T^*M$, sections of this bundle are differential forms of arbitrary degree. The Riemannian metric on TM induces a metric on this bundle by the formula
$$\langle \alpha, \beta \rangle := \sum_{\substack{i_1,\dots,i_k \\ j_i,\dots,j_k}} g^{i_1 j_1} \cdots g^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k}, \quad \text{for } \alpha, \beta \in \Omega^k(M).$$
Often it is useful to consider an orthonormal frame $\eta^1, \dots, \eta^n$ for $\Omega^1(M)$. Writing $\alpha = \alpha_{i_1 \dots i_k} \eta^{i_1} \wedge \dots \wedge \eta^{i_k}$ and similar for β we obtain
$$\langle \alpha, \beta \rangle = \sum_{i_1,\dots,i_k} \alpha_{i_1 \dots i_k} \beta_{i_1 \dots i_k}.$$

a) Given a vector field $X \in \mathfrak{X}(M)$, consider the operators

$$\iota_X \alpha, \quad \tilde{X} \wedge \alpha, \quad \text{for } \alpha \in \Omega^k(M)$$

Prove that these operators are adjoint to each other, i.e. show that

$$\langle \iota_X \alpha, \beta \rangle = \langle \alpha, \tilde{X} \wedge \beta \rangle$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$.

b) Use the previous exercise to prove that $\bigwedge T^*M$, equipped with the Levi-Civita connection and the action

$$\psi(X)\alpha := \tilde{X} \wedge \alpha - \iota_X \alpha, \quad \text{for } \alpha \in \Omega^k(M),$$

is a Clifford bundle.